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相关系数 $ARMA(p, q)$ 序列分析方法

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摘要: 本文提出相关系数 $ARMA(p, q)$ 序列分析方法。相关系数 $ARMA(p, q)$ 序列是从非平稳序列中分离出的一类工程上常见且便于研究的时间序列, 在模式识别、故障诊断、信号处理、自动控制和结构响应分析等领域有着广泛的应用。传统的相关函数 $ARMA(p, q)$ 序列仅是它的一个特例。文中建立了相关系数 $ARMA(p, q)$ 序列的条件极大似然估计和精确极大似然估计, 前者在样本较大时简单便于工程应用, 后者则在样本较小时仍具有较高的精度, 它们通过时域的全程分析, 充分利用样本信息确定相关系数 $ARMA(p, q)$ 序列的均值函数、方差函数和相关系数函数。在此基础上可进行高精度的频谱分析。

关键词: 航空、航天推进系统; 时间序列; $ARMA(p, q)$ 序列; 相关系数 $ARMA(p, q)$ 序列; 非平稳随机过程; 极大似然估计;

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Analysis Method of Correlation Coefficient $ARMA(p, q)$ Series

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Abstract: The correlation coefficient $ARMA(p, q)$ series of which mean and variance may vary with time is widely used in engineering, such as pattern recognition, fault diagnosis, signal processing, automatic control, structure response analysis and so on. The traditional correlation function stationary $ARMA(p, q)$ model is just a special case of that. The conditional maximum likelihood estimation (MLE) and exact MLE for the correlation coefficient $ARMA(p, q)$ series are established in this paper, which can obtain its mean function, variance function and correlation coefficient function with high precision. The conditional MLE is simple and convenient for engineering application when the sample is large enough, and the exact MLE is still precise in the case of small sample. Based on analyzing in time domain, the spectral analysis of correlation coefficient $ARMA(p, q)$ series can be done.

Key words: aerospace propulsion system; time series; $ARMA(p, q)$ series; correlation coefficient $ARMA(p, q)$ series; non-stationary random processes; maximum likelihood estimation;

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1 引言

文献[1]提出相关系数平稳过程的概念,文献[2]进一步建立相关系数平稳过程时频分析方法,提出相关系数 $AR(p)$ 、 $MA(q)$ 和 $ARMA(p, q)$ 序列的概念,给出相关系数 $AR(p)$ 序列的条件极大似然估计和精确极大似然估计方法,讨论了相关系数平稳过程(序列)的谱密度、随机项谱密度和趋势项谱密度,分析了线性系统对相关系数平稳过程输入的响应。本文进一步建立相关系数 $ARMA(p, q)$ 序列分析方法。相关系数 $ARMA(p, q)$ 序列是从非平稳序列中分离出来的一类工程上常见且便于研究的时间序列,其均值和方差都可随时间变化,与均值和方差为常数的相关函数 $ARMA(p, q)$ 序列相比,它能够更好地描述工程中的时间序列。文中给出相关系数 $ARMA(p, q)$ 序列的条件极大似然估计和精确极大似然估计方法。它们均能在时域进行全程分析,具有信息量大的优点,可以高精度地确定相关系数 $ARMA(p, q)$ 序列的均值函数、方差函数和相关系数函数。

2 相关系数 $ARMA(p, q)$ 序列条件极大似然估计方法

根据文献[2]中定义1可知,对于相关系数平稳过程 $X(t)$,令 $X_i = X(t_i)$, $t_i = t_0 + i\Delta t$, $i = 1, 2, \dots, n$, 其中 t_0 和 Δt 分别为采样初始时刻和采样间隔,则 $X = (X_1, X_2, \dots, X_n)$ 为相关系数平稳序列。再令 $Y_i = [X_i - \mu(a, t_i)]/I(b, t_i)$, 若 $Y = (Y_1, Y_2, \dots, Y_n)$ 为相关函数 $ARMA(p, q)$ 序列,则称 $X = (X_1, X_2, \dots, X_n)$ 为相关系数 $ARMA(p, q)$ 序列,其中 $E[X(t)] = \mu(a, t)$, $\text{Var}[X(t)] = \sigma^2 I^2(b, t)$; $a = (a_0, a_1, \dots, a_r)'$, $b = (b_0, b_1, \dots, b_s)$ 和 σ 均

为待定参数。本文将详细讨论相关系数 $ARMA(p, q)$ 序列的均值 $\mu(a, t)$, 方差 $\sigma^2 I^2(b, t)$ 和自相关系数 $\rho_k(k = 1, 2, \dots)$ 的确定。由相关函数 $ARMA(p, q)$ 序列理论可知^[3], Y 满足 $ARMA(p, q)$ 模型:

$$Y_t - \sum_{i=1}^p \eta Y_{t-i} = \epsilon_t - \sum_{j=1}^q \theta \epsilon_{t-j} \quad (1)$$

$$E[\epsilon_t \epsilon_{t-k}] = \begin{cases} 0 & k \neq 0 \\ \sigma_\epsilon^2 & k = 0 \end{cases} \quad (2)$$

设 $x_{ji} = x(t_{ji})$, $t_{ji} = t_{j0} + i\Delta t$, $i = 1, 2, \dots, n_j$, 是相关系数平稳正态过程 $X(t)$ 的第 j 个样本函数的一组观测值。 $x_j = (x_{j1}, x_{j2}, \dots, x_{jn_j})$ 为相关系数 $ARMA(p, q)$ 序列, $y_j = (y_{j1}, y_{j2}, \dots, y_{jn_j})$ 为相关函数 $ARMA(p, q)$ 序列, 其中 $y_{ji} = [x_{ji} - \mu(a, t_{ji})]/I(b, t_{ji})$, $j = 1, 2, \dots, m$ 。

首先讨论简单且便于工程应用的条件极大似然估计方法。根据文献[2]中定理2, 在每个样本的初始值 $y_{j(1-p)} = y_{j(2-p)} = \dots = y_{j0} = \epsilon_{j(1-q)} = \epsilon_{j(2-q)} = \dots = \epsilon_{j0} = 0$ ($j = 1, 2, \dots, m$)条件下, 由 $x_1, x_2, \dots, x_m$ 的联合密度函数可知, 参数 $a, b, \eta = (\eta_1, \eta_2, \dots, \eta_p)'$, $\theta = (\theta_1, \theta_2, \dots, \theta_q)$ 和 σ_ϵ 的条件对数似然函数为:

$$\ln L(a, b, \eta, \theta, \sigma_\epsilon) = - \sum_{j=1}^m \left[\frac{n_j - \lambda}{2} \ln(2\pi) + (n_j - \lambda) \ln \sigma_\epsilon + \sum_{i=\lambda+1}^{n_j} \frac{\epsilon_{ji}^2}{2\sigma_\epsilon^2} + \sum_{i=\lambda+1}^{n_j} \ln I(b, t_{ji}) \right] \quad (3)$$

式中 $\lambda = \max(p, q)$, ϵ_{ji} 可由下式递推得到:

$$\epsilon_{ji} = y_{ji} - \sum_{i=1}^p \eta y_{j(i-1)} + \sum_{j=1}^q \theta \epsilon_{j(i-1)} \quad (4)$$

因此, a, b, η, θ 和 σ_ϵ 的估计值可由下面极大似然方程组求得:

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_\epsilon)}{\partial a_k} = - \frac{1}{\sigma_\epsilon^2} \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \epsilon_{ji} \frac{\partial \epsilon_{ji}}{\partial a_k} = 0 \quad k = 0, 1, \dots, r \quad (5)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_\epsilon)}{\partial b_k} = - \frac{1}{\sigma_\epsilon^2} \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \epsilon_{ji} \frac{\partial \epsilon_{ji}}{\partial b_k} - \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \frac{1}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial b_k} = 0 \quad k = 0, 1, \dots, s \quad (6)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_\epsilon)}{\partial \eta_k} = - \frac{1}{\sigma_\epsilon^2} \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \epsilon_{ji} \frac{\partial \epsilon_{ji}}{\partial \eta_k} = 0 \quad k = 1, 2, \dots, p \quad (7)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_\epsilon)}{\partial \theta_k} = - \frac{1}{\sigma_\epsilon^2} \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \epsilon_{ji} \frac{\partial \epsilon_{ji}}{\partial \theta_k} = 0 \quad k = 1, 2, \dots, q \quad (8)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_\epsilon)}{\partial \sigma_\epsilon} = - \frac{1}{\sigma_\epsilon} \sum_{j=1}^m (n_j - \lambda) + \frac{1}{\sigma_\epsilon^3} \sum_{j=1}^m \sum_{i=\lambda+1}^{n_j} \epsilon_{ji}^2 = 0 \quad (9)$$

由于 $\frac{\partial \epsilon_{ji}}{\partial a_k} = \frac{\partial \epsilon_{ji}}{\partial b_k} = \frac{\partial \epsilon_{ji}}{\partial \eta_k} = \frac{\partial \epsilon_{ji}}{\partial \theta_k} = 0$, $i = 1 - q, 2 - q, \dots, 0$, 所以 $\frac{\partial \epsilon_{ji}}{\partial a_k}$, $\frac{\partial \epsilon_{ji}}{\partial b_k}$, $\frac{\partial \epsilon_{ji}}{\partial \eta_k}$ 和 $\frac{\partial \epsilon_{ji}}{\partial \theta_k}$, $i = 1, 2, \dots, n_j$, 可由下面

诸式递推求得:

$$\frac{\partial \epsilon_i}{\partial a_k} = - \frac{1}{I(b, t_{ji})} \frac{\partial u(a, t_{ji})}{\partial a_k} + \sum_{j=1}^p \frac{\eta}{I(b, t_{j(i-n)})} \frac{\partial u(a, t_{j(i-n)})}{\partial a_k} + \sum_{j=1}^q \theta \frac{\partial \epsilon_{j(i-n)}}{\partial a_k} \quad (10)$$

$$\frac{\partial \epsilon_i}{\partial b_k} = - \frac{y_{ji}}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial b_k} + \sum_{j=1}^p \frac{\eta y_{j(i-n)}}{I(b, t_{j(i-n)})} \frac{\partial I(b, t_{j(i-n)})}{\partial b_k} + \sum_{j=1}^q \theta \frac{\partial \epsilon_{j(i-n)}}{\partial b_k} \quad (11)$$

$$\frac{\partial \epsilon_i}{\partial \eta} = - y_{j(i-k)} + \sum_{j=1}^q \theta \frac{\partial \epsilon_{j(i-n)}}{\partial \eta} \quad (12)$$

$$\frac{\partial \epsilon_i}{\partial \theta} = \epsilon_{j(i-k)} + \sum_{j=1}^q \theta \frac{\partial \epsilon_{j(i-n)}}{\partial \theta} \quad (13)$$

式(9)可以进一步简化为:

$$\sigma_\epsilon^2 = \frac{1}{n-m} \lambda \sum_{j=1}^m \sum_{i=1}^{n_j} \epsilon_i^2 \quad (14)$$

式中

$$n = \sum_{j=1}^m n_j \quad (15)$$

联立式(5)~式(8)求解 a, b, η 和 θ 时需迭代计算, 若采用拟牛顿迭代法、梯度法等传统计算方法, 则往往容易发散或收敛很慢, 甚至导致不同的初值, 收敛于不同的解。所以, 建议采用文献[4]中方法求解。

式(3)中, 受条件 $y_{j(1-p)} = y_{j(2-p)} = \dots = y_{j0} = \epsilon_{j(1-q)} = \epsilon_{j(2-q)} = \dots = \epsilon_{j0} = 0$ 影响较大的每个样本前 λ 项 $\epsilon_1, \epsilon_2, \dots, \epsilon_\lambda$ 已被剔除。当样本较大时, 该条件对式(3)给出的对数似然函数影响很小, 可以忽略不计, 因此, 条件极大似然估计具有相当高的精度, 完全可以满足工程上的需要。

3 自相关系数计算方法

求得 η, θ 和 σ_ϵ 后, 进一步由下面诸式计算

ARMA(p, q) 序列相关系数 $\rho_k (k = 1, 2, \dots)$ 和 σ_ϵ^2

$$\rho_k = Y_k / Y_0 \quad k = 1, 2, \dots \quad (16)$$

$$\sigma_\epsilon^2 = Y_0 \quad (17)$$

式中

$$Y_0 = l_{00} + \sum_{i=1}^p \eta Y_i \quad (18)$$

$$(Y_1, Y_2, \dots, Y_p)' = L Y L' \quad (19)$$

$$Y_k = \sum_{i=1}^p \eta Y_{k-i} + w_k \quad k = p+1, p+2, \dots \quad (20)$$

$$L Y = (l_{ij})_{p \times p} \quad (21)$$

$$L Y = (l_{01}, l_{02}, \dots, l_{0p})' \quad (22)$$

$$l_{ii} = \eta_{i-1} + \eta_{i+1} + \eta \eta \quad (23)$$

$$l_{0i} = \begin{cases} \sum_{j=1}^q \theta G_{i-j} \sigma_\epsilon^2 - \eta l_{00} & i \leq q \\ -\eta l_{00} & i > q \end{cases} \quad (24)$$

$$l_{00} = \left(1 - \sum_{j=1}^q \theta G_j \right) \sigma_\epsilon^2 \quad (25)$$

$$w_k = \begin{cases} - \sum_{j=k}^q \theta G_{i-k} \sigma_\epsilon^2 & k \leq q \\ 0 & k > q \end{cases} \quad (26)$$

式中 $\eta = \theta = -1$; 当 $i > p$ 或 $i < 0$ 时, $\eta = 0$; 当 $i > q$ 时, $\theta = 0$ 。 G_i 为格林函数, 可通过下式递推得到 ($G_0 = 1$):

$$G_i = -\theta + \sum_{j=1}^i \eta G_{i-j} \quad i = 1, 2, \dots \quad (27)$$

相关系数 ARMA(p, q) 过程的自相关系数 $\rho_k, k = 1, 2, \dots$, 也可以通过总体谱^[3]求得:

$$\sigma_\epsilon^2 = \int_{-\pi}^{\pi} S_Y(\omega) d\omega \quad (28)$$

$$\rho_k = \frac{1}{\sigma_\epsilon^2} \int_{-\pi}^{\pi} S_Y(\omega) e^{j\omega k} d\omega \quad k = 1, 2, \dots \quad (29)$$

式中

$$S_Y(\omega) = \frac{\sigma_\epsilon^2}{2\pi} \frac{1 - \theta e^{j\omega} - \theta e^{j2\omega} - \dots - \theta e^{jq\omega}}{1 - \eta e^{-j\omega} - \eta e^{-j2\omega} - \dots - \eta e^{-jp\omega}} \times \frac{1 - \theta e^{-j\omega} - \theta e^{-j2\omega} - \dots - \theta e^{-jq\omega}}{1 - \eta e^{-j\omega} - \eta e^{-j2\omega} - \dots - \eta e^{-jp\omega}} \quad (30)$$

式(28)~式(30)中的 j 为虚数单位。

此外, ARMA(p, q) 模型的阶数可采用似然比检验^[1]、AIC 准则^[5]或偏相关系数准则^[5]确定。

4 相关系数ARMA(p, q)序列精确极大似然估计方法

当样本量较小时, 条件极大似然估计的误差往往较大, 为此, 下面建立相关系数 ARMA(p, q) 序列精确极大似然估计方法。

4.1 相关系数ARMA(p, q)精确似然函数

根据文献[2, 6], 参数 a, b, η, θ 和 σ_ϵ 的精确对数似然函数为:

$$\ln L(a, b, \eta, \theta, \sigma_\epsilon) = - \sum_{j=1}^m \left[\frac{n_j}{2} \ln(2\pi) + n_j \ln \sigma_\epsilon + \frac{1}{2} \ln |\Omega| + \frac{1}{2} \ln |D_j| + \frac{1}{2\sigma_\epsilon^2} \epsilon_j' \left(\frac{1}{2\sigma_\epsilon^2} h_j D_j^{-1} h_j + \sum_{j=1}^{n_j} \ln I(b, t_{ji}) \right) \right] \quad (31)$$

式中

$$\epsilon_j = (\epsilon_{j1}, \epsilon_{j2}, \dots, \epsilon_{jn_j})' \quad (32)$$

$$h_j = (h_{j1}, h_{j2}, \dots, h_{j(p+q)})' \quad (33)$$

$$\Omega = \begin{bmatrix} \sigma_{\epsilon}^2 \Gamma & C \\ C & I_q \end{bmatrix} \quad (34)$$

$$D_j = \Omega^{-1} + F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 F_j \quad (35)$$

式中 ϵ 和 h_j 的元素可由下式递推得到:

$$\epsilon_{ji} = y_{ji} - \sum_{l=1}^p \eta_{y_{j(i-l)}} + \sum_{l=1}^q \theta_l \epsilon_{j(i-l)} \quad (36)$$

$$h_{ji} = \begin{cases} \sum_{u=1}^i \sum_{v=0}^{n_j-u} \alpha_{p-i+u} \epsilon_{j(u+v)} & i \leq p \\ - \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \alpha_{p+q-i+u} \epsilon_{j(u+v)} & i > p \end{cases} \quad (37)$$

$$\alpha_i = \sum_{l=1}^i \theta_l \alpha_{i-l} \quad i = 1, 2, \dots, n_j - 1 \quad (38)$$

式中 $y_{ji} = 0 (i = 1-p, 2-p, \dots, 0)$, $\epsilon_{ji} = 0 (i = 1-q, 2-q, \dots, 0)$ 和 $\alpha_0 = 1$. I_q 为 $q \times q$ 阶单位矩阵, $\Gamma = (\gamma_{i-u+1})_{p \times p}$, 其中 $\gamma_i (i = 0, 1, \dots, p-1)$ 由式

4.2 相关系数 ARMA(p, q) 序列精确极大似然估计

对式(31)求偏导, 得到求解参数 a, b, η, θ 和 σ_{ϵ}^2 的极大似然方程组:

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_{\epsilon}^2)}{\partial a_k} = - \frac{1}{\sigma_{\epsilon}^2} \sum_{j=1}^m \left(\epsilon_j \frac{\partial \epsilon_j}{\partial a_k} - h_j D_j^{-1} \frac{\partial h_j}{\partial a_k} \right) = 0 \quad k = 0, 1, \dots, r \quad (43)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_{\epsilon}^2)}{\partial b_k} = - \frac{1}{\sigma_{\epsilon}^2} \sum_{j=1}^m \left(\epsilon_j \frac{\partial \epsilon_j}{\partial b_k} - h_j D_j^{-1} \frac{\partial h_j}{\partial b_k} \right) - \sum_{i=1}^m \sum_{j=1}^{n_j} \frac{1}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial b_k} = 0 \quad k = 0, 1, \dots, s \quad (44)$$

$$\begin{aligned} \frac{\partial \ln L(a, b, \eta, \theta, \sigma_{\epsilon}^2)}{\partial \eta} &= - \frac{m}{2|\Omega|} \sum_{i=1}^{p+q} \Omega_{ii} \frac{\partial \Omega_{ii}}{\partial \eta} - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^{p+q} \frac{D_{jii}}{|D_j|} \frac{\partial D_{jii}}{\partial \eta} - \\ &\frac{1}{2\sigma_{\epsilon}^2} \sum_{j=1}^m \left(2\epsilon_j \frac{\partial \epsilon_j}{\partial \eta} + h_j D_j^{-1} \frac{\partial h_j}{\partial \eta} D_j^{-1} h_j - 2h_j D_j^{-1} \frac{\partial h_j}{\partial \eta} \right) = 0 \quad k = 1, 2, \dots, p \end{aligned} \quad (45)$$

$$\begin{aligned} \frac{\partial \ln L(a, b, \eta, \theta, \sigma_{\epsilon}^2)}{\partial \theta} &= - \frac{m}{2|\Omega|} \sum_{i=1}^{p+q} \Omega_{ii} \frac{\partial \Omega_{ii}}{\partial \theta} - \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^{p+q} \frac{D_{jii}}{|D_j|} \frac{\partial D_{jii}}{\partial \theta} - \\ &\frac{1}{2\sigma_{\epsilon}^2} \sum_{j=1}^m \left(2\epsilon_j \frac{\partial \epsilon_j}{\partial \theta} + h_j D_j^{-1} \frac{\partial h_j}{\partial \theta} D_j^{-1} h_j - 2h_j D_j^{-1} \frac{\partial h_j}{\partial \theta} \right) = 0 \quad k = 1, 2, \dots, q \end{aligned} \quad (46)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \sigma_{\epsilon}^2)}{\partial \sigma_{\epsilon}^2} = - \frac{1}{\sigma_{\epsilon}^2} \sum_{j=1}^m n_j + \frac{1}{\sigma_{\epsilon}^3} \sum_{j=1}^m (\epsilon_j^2 - h_j D_j^{-1} h_j) = 0 \quad (47)$$

式(47)可简化为:

$$\sigma_{\epsilon}^2 = \frac{1}{n} \sum_{j=1}^m (\epsilon_j^2 - h_j D_j^{-1} h_j) \quad (48)$$

式中 n 由式(15)给出:

$$\frac{\partial \epsilon_{ji}}{\partial a_k} = - \frac{1}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial a_k} + \sum_{l=1}^p \frac{\eta_{y_{j(i-l)}}}{I(b, t_{j(i-l)})} \frac{\partial I(b, t_{j(i-l)})}{\partial a_k} + \sum_{l=1}^q \theta_l \frac{\partial \epsilon_{j(i-l)}}{\partial a_k} \quad (49)$$

$$\frac{\partial \epsilon_{ji}}{\partial b_k} = - \frac{y_{ji}}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial b_k} + \sum_{l=1}^p \frac{\eta_{y_{j(i-l)}}}{I(b, t_{j(i-l)})} \frac{\partial I(b, t_{j(i-l)})}{\partial b_k} + \sum_{l=1}^q \theta_l \frac{\partial \epsilon_{j(i-l)}}{\partial b_k} \quad (50)$$

$$\frac{\partial h_j}{\partial \eta} = - \Omega^{-1} \frac{\partial \Omega}{\partial \eta} \Omega^{-1} + F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial F_j}{\partial \eta} + \left[F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial F_j}{\partial \eta} \right]' \quad (51)$$

$$\begin{aligned} \frac{\partial h_j}{\partial \theta} &= - \Omega^{-1} \frac{\partial \Omega}{\partial \theta} \Omega^{-1} - F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial L \hat{\theta}_j^1}{\partial \theta} F_j - \left[F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial L \hat{\theta}_j^1}{\partial \theta} F_j \right]' + \\ &F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial F_j}{\partial \theta} + \left[F_j (L \hat{\theta}_j^1) L' \hat{\theta}_j^1 \frac{\partial F_j}{\partial \theta} \right]' \end{aligned} \quad (52)$$

(18) 和式(19)计算。 $L \hat{\theta}_j = (l_{\theta_{uv}})_{n_j \times n_j}$, $L \hat{\theta}_j^1 = (l_{\theta_{uv}}^1)_{n_j \times n_j}$, $F_j = (f_{uv})_{n_j \times (p+q)}$ 和 $C = (c_{uv})_{q \times p}$ 的元素分别为:

$$l_{\theta_{uv}} = \begin{cases} - \theta_{u-v} & u \geq v \\ 0 & u < v \end{cases} \quad (39)$$

$$l_{\theta_{uv}}^1 = \begin{cases} \alpha_{u-v} & u \geq v \\ 0 & u < v \end{cases} \quad (40)$$

$$f_{juv} = \begin{cases} \eta_{p+u-v} & 1 \leq u \leq v \leq p \\ - \theta_{p+q+u-v} & 1 \leq u \leq v-p \leq q \\ 0 & \text{其余} \end{cases} \quad (41)$$

$$c_{uv} = \begin{cases} G_{q-p-u+v} & q-p-u+v \geq 0 \\ 0 & q-p-u+v < 0 \end{cases} \quad (42)$$

其中 G_i 为格林函数, 由式(27)给出。

$$\frac{\partial h_{ij}}{\partial \eta} = \begin{cases} \sum_{v=0}^{n_j+p-k-i} \alpha \epsilon_{(i+k+v-p)} \delta(k \geq p-i+1) + \sum_{u=1}^i \sum_{v=0}^{n_j-u} \alpha \eta_{p-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial \eta} & i \leq p \\ - \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \alpha \theta_{p+q-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial \eta} & i > p \end{cases} \quad (53)$$

$$\frac{\partial h_{ij}}{\partial \alpha} = \begin{cases} \sum_{u=1}^i \sum_{v=0}^{n_j-u} \eta_{p-i+u} \epsilon_{(u+v)} \frac{\partial \alpha}{\partial \alpha} + \sum_{u=1}^i \sum_{v=0}^{n_j-u} \alpha \eta_{p-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial \alpha} & i \leq p \\ - \sum_{v=0}^{n_j+p+q-k-i} \alpha \epsilon_{(i+k+v-p-q)} \delta(k \geq p+q-i+1) - \\ \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \theta_{p+q-i+u} \epsilon_{(u+v)} \frac{\partial \alpha}{\partial \alpha} - \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \alpha \theta_{p+q-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial \alpha} & i > p \end{cases} \quad (54)$$

$$\frac{\partial h_{ij}}{\partial a_k} = \begin{cases} \sum_{u=1}^i \sum_{v=0}^{n_j-u} \alpha \eta_{p-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial a_k} & i \leq p \\ - \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \alpha \theta_{p+q-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial a_k} & i > p \end{cases} \quad (55)$$

$$\frac{\partial h_{ij}}{\partial b_k} = \begin{cases} \sum_{u=1}^i \sum_{v=0}^{n_j-u} \alpha \eta_{p-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial b_k} & i \leq p \\ - \sum_{u=1}^{i-p} \sum_{v=0}^{n_j-u} \alpha \theta_{p+q-i+u} \frac{\partial \epsilon_{(u+v)}}{\partial b_k} & i > p \end{cases} \quad (56)$$

$$\frac{\partial \epsilon_i}{\partial \eta} = -y_{j(i-k)} + \sum_{l=1}^q \theta_l \frac{\partial \epsilon_{(i-l)}}{\partial \eta} \quad (57)$$

$$\frac{\partial \epsilon_i}{\partial \alpha} = \epsilon_{(i-k)} + \sum_{l=1}^q \theta_l \frac{\partial \epsilon_{(i-l)}}{\partial \alpha} \quad (58)$$

$$\frac{\partial \omega_i}{\partial \eta} = \begin{cases} \frac{1}{\sigma_\epsilon^2} \frac{\partial y_{(i-l)}}{\partial \eta} & 1 \leq l \leq i \leq p \\ \frac{\partial \epsilon_{(i-p+l)}}{\partial \eta} & p+1 \leq i \leq p+q, 1 \leq l \leq p \\ 0 & p+1 \leq l \leq i \leq p+q \\ \frac{\partial \omega_i}{\partial \eta} & i < l \end{cases} \quad (59)$$

$$\frac{\partial \omega_i}{\partial \alpha} = \begin{cases} \frac{1}{\sigma_\epsilon^2} \frac{\partial y_{(i-l)}}{\partial \alpha} & 1 \leq l \leq i \leq p \\ \frac{\partial \epsilon_{(i-p+l)}}{\partial \alpha} & p+1 \leq i \leq p+q, 1 \leq l \leq p \\ 0 & p+1 \leq l \leq i \leq p+q \\ \frac{\partial \omega_i}{\partial \alpha} & i < l \end{cases} \quad (60)$$

式中 $\frac{\partial \epsilon_i}{\partial \eta} = \frac{\partial \epsilon_i}{\partial \alpha} = \frac{\partial \omega_i}{\partial \eta} = \frac{\partial \omega_i}{\partial \alpha} = 0$ ($i = 1-q, 2-q, \dots, 0$); ω_l , Ω_l 和 d_{jil} , D_{jil} 分别为矩阵 Ω 和 D_j 的元素及其代数余子式; $\delta[g(x)]$ 定义为: 当 $g(x)$ 成立时 δ

$[g(x)] = 1$, 否则 $\delta[g(x)] = 0$; $\frac{\partial y}{\partial \eta}$ 和 $\frac{\partial y}{\partial \alpha}$ 分别由下面两式计算:

$$\frac{\partial (y_1, y_2, \dots, y_p)}{\partial \eta} = -L^{-1} \frac{\partial L}{\partial \eta} L^{-1} y + L^{-1} \frac{\partial y}{\partial \eta} \quad (61)$$

$$\frac{\partial (y_1, y_2, \dots, y_p)}{\partial \alpha} = L^{-1} \frac{\partial L}{\partial \alpha} y \quad (62)$$

4.3 相关系数ARMA (1, 1) 精确极大似然估计

为了便于理解, 同时也由于相关系数ARMA (1, 1) 序列是工程中最常见的情况, 下面对相关系数ARMA (1, 1) 序列的精确极大似然估计进行专门的讨论。此时式(1)可简化为:

$$Y_t - \eta Y_{t-1} = \epsilon_t - \theta \epsilon_{t-1} \quad (63)$$

参数 a, b, η, θ 和 σ_ϵ 的对数似然函数为:

$$\ln L(a, b, \eta, \theta, \sigma_\epsilon) = - \sum_{j=1}^m \left[\frac{n_j}{2} \ln(2\pi) + n_j \ln \sigma_\epsilon + \frac{1}{2} \ln |\Omega| + \frac{1}{2} \ln |D_j| + \frac{1}{2\sigma_\epsilon^2} S_j(a, b, \eta, \theta) + \sum_{l=1}^{n_j} \ln I(b, t_{jl}) \right] \quad (64)$$

$$\text{式中 } |\Omega| = \frac{(\eta - \theta)^2}{1 - \eta^2} \quad (65)$$

$$|D_j| = \frac{1 - \eta}{(\eta - \theta)^2} + \frac{1 - \theta^{n_j}}{1 - \theta^2} \quad (66)$$

$$S_j(a, b, \eta, \theta) = \sum_{l=1}^{n_j} \left[\sum_{i=1}^i \theta^{i-1} (y_{jl} - \eta y_{j(i-1)}) \right]^2 + \frac{(\theta^2 - 1)(\theta - \eta)^2}{(1 - \eta)^2 - \theta^{n_j}(\theta - \eta)^2} \times \left[\sum_{l=1}^{n_j} \sum_{i=1}^i \theta^{i-1} (y_{jl} - \eta y_{j(i-1)}) \right]^2 \quad (67)$$

其中, $y_{j0} = 0$ 。对式(64)求偏导, 得到参数 a, b, η, θ 和 σ_ϵ 的极大似然方程组:

$$\frac{\partial \ln L(a, b, \eta, \theta, \alpha_k)}{\partial a_k} = - \frac{1}{2\sigma_k^2} \sum_{j=1}^m \frac{\partial \ln L(a, b, \eta, \theta)}{\partial a_k} = 0 \quad k = 0, 1, \dots, r \quad (68)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \alpha_k)}{\partial b_k} = - \frac{1}{2\sigma_k^2} \sum_{j=1}^m \frac{\partial \ln L(a, b, \eta, \theta)}{\partial b_k} - \sum_{j=1}^m \sum_{i=1}^{n_j} \frac{1}{I(b, t_{ji})} \frac{\partial I(b, t_{ji})}{\partial b_k} = 0 \quad k = 0, 1, \dots, s \quad (69)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \alpha_k)}{\partial \eta} = - \frac{m}{2|\Omega|} \frac{\partial |\Omega|}{\partial \eta} - \sum_{j=1}^m \frac{1}{2|D_j|} \frac{\partial |D_j|}{\partial \eta} - \frac{1}{2\sigma_k^2} \sum_{j=1}^m \frac{\partial \ln L(a, b, \eta, \theta)}{\partial \eta} = 0 \quad (70)$$

$$\frac{\partial \ln L(a, b, \eta, \theta, \alpha_k)}{\partial \theta} = - \frac{m}{2|\Omega|} \frac{\partial |\Omega|}{\partial \theta} - \sum_{j=1}^m \frac{1}{2|D_j|} \frac{\partial |D_j|}{\partial \theta} - \frac{1}{2\sigma_k^2} \sum_{j=1}^m \frac{\partial \ln L(a, b, \eta, \theta)}{\partial \theta} = 0 \quad (71)$$

$$\alpha_k^2 = \frac{1}{n} \sum_{j=1}^m S_j(a, b, \eta, \theta) \quad (72)$$

式中 n 由式(15)给出;

$$\begin{aligned} \frac{\partial \ln L(a, b, \eta, \theta)}{\partial \eta} = & -2 \sum_{j=1}^{n_j} \left[\sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \sum_{i=1}^i \theta^{i-1} y_{j(l-1)} \right] + \\ & \frac{2(\theta^2 - 1)^2 (\theta - \eta) (1 - \eta)}{(1 - \eta)^2 - \theta^{n_j} (\theta - \eta)^2} \left[\sum_{i=1}^{n_j} \sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \right]^2 - \\ & \frac{2(\theta^2 - 1)(\theta - \eta)^2}{(1 - \eta)^2 - \theta^{n_j} (\theta - \eta)^2} \sum_{j=1}^{n_j} \sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \sum_{i=1}^{n_j} \sum_{i=1}^i \theta^{i-1} y_{j(l-1)} \\ \frac{\partial \ln L(a, b, \eta, \theta)}{\partial \theta} = & 2 \sum_{j=1}^{n_j} \left[\sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \sum_{i=1}^i (i-1) \theta^{i-2} (y_{ji} - \eta_{j(l-1)}) \right] + \\ & \left[\sum_{i=1}^{n_j} \sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \right]^2 \times \\ & \left\{ \frac{2\theta^{n_j} [(n_j - 1)\theta^2 - n_j] (\theta - \eta)^4 - 2\theta(\theta - \eta)(1 - \eta)(1 - \eta - 2\theta^2 + \eta\theta + \eta\theta)}{\theta(1 - \eta)^2 - \theta^{n_j} (\theta - \eta)^2} \right\} + \\ & \frac{2(\theta^2 - 1)(\theta - \eta)^2}{(1 - \eta)^2 - \theta^{n_j} (\theta - \eta)^2} \sum_{j=1}^{n_j} \sum_{i=1}^i \theta^{i-1} (y_{ji} - \eta_{j(l-1)}) \sum_{i=1}^{n_j} \sum_{i=1}^i (2i-1) \theta^{i-2} (y_{ji} - \eta_{j(l-1)}) \end{aligned} \quad (74)$$

5 结 论

(1) 相关系数 $ARMA(p, q)$ 序列是工程中常见的一类时间序列, 其均值和方差通常随时间而变化, 与均值和方差为常数的相关函数 $ARMA(p, q)$ 序列相比, 它能够更好地描述工程中的时间序列。传统的 $ARMA(p, q)$ 序列仅是它的一个特例。

(2) 建立了相关系数 $ARMA(p, q)$ 序列分析方法, 该方法可以在时域进行全程分析, 具有信息量大的优点, 能够高精度地确定相关系数 $ARMA(p, q)$ 序列的均值函数、方差函数和相关系数函数。

(3) 相关系数 $ARMA(p, q)$ 序列的条件极大似然估计在样本较大时具有相当高的精度, 而且简单便于工程应用。

(4) 相关系数 $ARMA(p, q)$ 序列精确极大似然估计在样本较小时仍具有精度高的优点。

(5) 采用本文方法求得相关系数 $ARMA(p,$

$q)$ 序列的均值函数、方差函数和相关系数函数后, 再结合文献[2]中方法可以对相关系数 $ARMA(p, q)$ 序列进行高精度的频谱分析。

(6) 本文方法不但对各态历经的相关系数 $ARMA(p, q)$ 序列适用, 而且对非各态历经的相关系数 $ARMA(p, q)$ 序列也适用, 前者只需 $m \geq 1$, 后者则要求 $m \geq 2$ 。

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